Pushable chromatic number of graphs with degree constraints*

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Abstract

Pushable homomorphisms and the pushable chromatic number χ_p of oriented graphs were introduced by Klostermeyer and MacGillivray in 2004. They notably observed that, for any oriented graph $\overrightarrow{G}$, we have $\chi_p(\overrightarrow{G}) \leq \chi_o(\overrightarrow{G}) \leq 2\chi_p(\overrightarrow{G})$, where $\chi_o(\overrightarrow{G})$ denotes the oriented chromatic number of $\overrightarrow{G}$. This stands as the first general bounds on χ_p. This parameter was further studied in later works.

This work is dedicated to the pushable chromatic number of oriented graphs fulfilling particular degree conditions. For all $\Delta \geq 29$, we first prove that the maximum value of the pushable chromatic number of a connected oriented graph with maximum degree Δ lies between $2^{\frac{\Delta}{2}-1}$ and $(\Delta-3)\cdot(\Delta-1)\cdot 2^{\Delta-1}+2$ which implies an improved bound on the oriented chromatic number of the same family of graphs. For subcubic oriented graphs, that is, when $\Delta \leq 3$, we then prove that the maximum value of the pushable chromatic number is 6 or 7. We also prove that the maximum value of the pushable chromatic number of oriented graphs with maximum average degree less than 3 lies between 5 and 6. The former upper bound of 7 also holds as an upper bound on the pushable chromatic number of planar oriented graphs with girth at least 6.

Keywords: oriented coloring, push operation, graph homomorphism, maximum degree, subcubic graph.

1 Introduction and main results

An *oriented graph* is a loopless directed graph without opposite arcs. Equivalently, an oriented graph $\overrightarrow{G}$ can be seen as an *orientation* of a simple undirected graph G. We always refer to an oriented graph $\overrightarrow{G}$ using an arrow symbol, which makes apparent that $\overrightarrow{G}$ is an orientation of G. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively, while we denote by $V(\overrightarrow{G})$ and $A(\overrightarrow{G})$ the sets of vertices and arcs of $\overrightarrow{G}$, respectively. Also, when referring to a notation, notion or term for $\overrightarrow{G}$ that is usually defined for undirected graphs, we implicitly refer to the corresponding notation, notion or term regarding G.

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The notions of *oriented coloring* and *oriented chromatic number* of oriented graphs were introduced by Courcelle [2] in 1994, and have been intensively studied since then (see the recent survey [19] for more details). One way of defining these notions is through the notion of *graph homomorphisms*. For two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$, a *homomorphism* from $\overrightarrow{G}$ to $\overrightarrow{H}$ is a mapping $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that $uv \in A(\overrightarrow{G})$ implies $\phi(u)\phi(v) \in A(\overrightarrow{H})$. We write $\overrightarrow{G} \to \overrightarrow{H}$ whenever a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$ exists. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the minimum order (number of vertices) of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G} \to \overrightarrow{H}$.

In 2004, Klostermeyer and MacGillivray [9] introduced the *pushable chromatic number* of oriented graphs. *Pushing* a vertex v of an oriented graph $\overrightarrow{G}$ means changing the orientation of all arcs incident with v, i.e., replacing every arc vu by the arc uv, and vice versa. Two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{G}'$ are *in a push relationship* if $\overrightarrow{G}'$ can be obtained from $\overrightarrow{G}$ by pushing some vertices of $\overrightarrow{G}$. Note that being in push relationship is an equivalence relation. The class of the oriented graphs that are in a push relationship with $\overrightarrow{G}$ is denoted by $[\overrightarrow{G}]$. Observe that any two oriented graphs from $[\overrightarrow{G}]$ have the same underlying graph, which is G.

For two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$, a *pushable homomorphism* from $\overrightarrow{G}$ to $\overrightarrow{H}$ is a mapping $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that there exists $\overrightarrow{G}' \in [\overrightarrow{G}]$ for which ϕ is a homomorphism from $\overrightarrow{G}'$ to $\overrightarrow{H}$. We write $\overrightarrow{G} \xrightarrow{\text{push}} \overrightarrow{H}$ whenever there exists a pushable homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. The *pushable chromatic number* $\chi_p(\overrightarrow{G})$ of $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G} \xrightarrow{\text{push}} \overrightarrow{H}$.

The seminal work of Klostermeyer and MacGillivray on these notions opened the way to more works on the topic. For instance, results on the pushable chromatic number can be found in [1, 7, 18], while the push operation was further studied in [6, 8, 10, 12, 14, 15, 16, 17]. Some complexity issues related to pushable homomorphisms were studied in [7, 9]. Regarding our investigations in this paper, an important result from the seminal work [9] of Klostermeyer and MacGillivray is the following general relation between χ_o and χ_p.

Theorem 1.1 (Klostermeyer, MacGillivray [9])**.** *For every oriented graph $\overrightarrow{G}$, we have*

$$\chi_p(\overrightarrow{G}) \leq \chi_o(\overrightarrow{G}) \leq 2\chi_p(\overrightarrow{G}).$$

Theorem 1.1 yields another point for studying the pushable chromatic number of oriented graphs, as it is a way to get bounds on the oriented chromatic number. Sen, in [18], also established a strong connection between pushable homomorphisms and oriented homomorphisms of oriented graphs.

The notions of oriented chromatic number and pushable chromatic number can also be extended to undirected graphs G by setting

$$\chi_o(G) = \max\{\chi_o(\overrightarrow{G}) : \overrightarrow{G} \text{ is an orientation of } G\}$$

and

$$\chi_p(G) = \max\{\chi_p(\overrightarrow{G}) : \overrightarrow{G} \text{ is an orientation of } G\}.$$

A natural question is, given a family $\mathcal{F}$ of undirected graphs, how large can the oriented chromatic number and the pushable chromatic number of its members be? In other words, we are interested in the two parameters $\chi_o(\mathcal{F}) = \max\{\chi_o(G) : G \in \mathcal{F}\}$ and $\chi_p(\mathcal{F}) = \max\{\chi_p(G) : G \in \mathcal{F}\}$. Regarding the pushable chromatic number, partial results were obtained for the families of outerplanar graphs, 2-trees, planar graphs, planar graphs with girth restrictions, and graphs with bounded acyclic chromatic number (see [7, 9, 18]). However, to the best of our knowledge, nothing general is known regarding the family $\mathcal{G}_\Delta$ of graphs with maximum degree Δ and the family $\mathcal{G}_\Delta^{\mathrm{c}}$ of connected graphs with maximum degree Δ. Unlike the ordinary chromatic number, the oriented and pushable chromatic number for the families $\mathcal{G}_\Delta$ and $\mathcal{G}_\Delta^{\mathrm{c}}$ can be different. A good illustration of this, is the fact that if two oriented graphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ individually admit a homomorphism to an oriented graph with order k, then it is not true that their disjoint union $\overrightarrow{G_1} + \overrightarrow{G_2}$ always does (consider, for instance, $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ being two tournaments on k

vertices that are not in push relashionship). Finding the oriented and pushable chromatic number of $\mathcal{G}^c_\Delta$ is our main concern in this paper.

We thus initiate the study of the pushable chromatic number of $\mathcal{G}^c_\Delta$. Adapting a probabilistic proof used by Kostochka, Sopena and Zhu in [11], we first provide general bounds for large enough Δ.

Theorem 1.2. *For all $\Delta \geq 29$, we have*

$$2^{\frac{\Delta}{2}-1} \leq \chi_p(\mathcal{G}^c_\Delta) \leq (\Delta-3)\cdot(\Delta-1)\cdot 2^{\Delta-1}+2.$$

Note that the lower bound and the upper bound in Theorem 1.2 are both exponential in Δ. Also, it is worth mentioning that the upper bound established in Theorem 1.2 is better than the upper bound that one would directly get from Theorem 1.1 and the best upper bound on $\chi_o(\mathcal{G}_\Delta)$ to date, which is that $\chi_o(\mathcal{G}_\Delta) \leq 2\Delta^2 \cdot 2^\Delta$ (see [11]). Actually, employing another trick used by Duffy in [3], Theorem 1.2 also yields the following improved upper bound on $\chi_o(\mathcal{G}^c_\Delta)$ as a side result.

Theorem 1.3. *For all $\Delta \geq 29$, we have*

$$2^{\frac{\Delta}{2}} \leq \chi_o(\mathcal{G}^c_\Delta) \leq (\Delta-3)\cdot(\Delta-1)\cdot 2^{\Delta}+2.$$

When it comes to coloring graphs with given maximum degree, a natural step to make is considering graphs with low maximum degree. This concern is actually a major one regarding oriented coloring, as it is still open what the value of $\chi_o(\mathcal{G}^c_3)$ is. Sopena [19] conjectured that $\chi_o(\mathcal{G}^c_3) = 7$, and, to date, we know that $7 \leq \chi_o(\mathcal{G}_3) \leq 9$ and $7 \leq \chi_o(\mathcal{G}^c_3) \leq 8$ hold (see [5, 19] and [4], respectively).

Due to the general connection between the oriented chromatic number and the pushable chromatic number, it makes sense wondering about $\chi_p(\mathcal{G}^c_3)$ as well. In this work, we provide the following result as a first step towards this question.

Theorem 1.4. *We have $6 \leq \chi_p(\mathcal{G}^c_3) \leq \chi_p(\mathcal{G}_3) \leq 7$.*

In graph coloring theory, another relevant aspect related to the vertex degrees is the maximum average degree. Precisely, the *maximum average degree* $\mathrm{mad}(G)$ of a graph G is

$$\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \text{ is a subgraph of } G\right\}.$$

In this work, we also study the pushable chromatic number of the family $\mathcal{G}^{\mathrm{mad}}_3 = \{G : \mathrm{mad}(G) < 3\}$ of graphs with maximum average degree less than 3. Our main result reads as follows.

Theorem 1.5. *We have $5 \leq \chi_p(\mathcal{G}^{\mathrm{mad}}_3) \leq 7$.*

It was previously proved in [18] that for the family $\mathcal{G}^{\mathrm{mad}}_{8/3} = \{G : \mathrm{mad}(G) < \frac{8}{3}\}$ we have $\chi_p(\mathcal{G}^{\mathrm{mad}}_{8/3}) = 4$. More precisely, in that result the equality follows from the existence of planar graphs with girth 8 and pushable chromatic number 4. This, and, because planar graphs with girth at least 6 have maximum average degree strictly less than 3, Theorem 1.5 yields the following, where $\mathcal{P}_6$ denotes the family of planar graphs with girth at least 6.

Theorem 1.6. *We have $4 \leq \chi_p(\mathcal{P}_6) \leq 7$.*

This paper is organized as follows. We start off by introducing, in Section 2, some notation, terminology, and preliminary results. The next sections are devoted to proving Theorems 1.2 and 1.3 (Section 3), Theorem 1.4 (Section 4), and Theorem 1.5 (Section 5). Open questions and perspectives for future work are discussed in Section 6.

2 Notation, terminology, and preliminary results

For an arc uv of an oriented graph $\overrightarrow{G}$, we say that u is a *$-$-neighbor* of v while v is a *$+$-neighbor* of u. The set of the $-$-neighbors ($+$-neighbors, respectively) of any vertex v of $\overrightarrow{G}$ is denoted by $N^-(v)$ ($N^+(v)$, respectively). For a set S of vertices of $\overrightarrow{G}$ and some $\alpha \in \{-,+\}$, we define $N^\alpha(S) = \bigcup_{v \in S} N^\alpha(v)$.

To prove that all oriented graphs from a given family admit homomorphisms to a given oriented graph $\overrightarrow{H}$, we generally need $\overrightarrow{H}$ to have very strong properties. In most of the proofs from the literature on the topic, and in our proofs in the current paper as well, a strong property we consider is the possibility, given a partial homomorphism from an oriented graph $\overrightarrow{G}$ to $\overrightarrow{H}$, to extend the partial homomorphism to another vertex v of $\overrightarrow{G}$, assuming some of its neighbors (which can be in any of $N^-(v)$ and $N^+(v)$) have already been assigned an image. A way to define this intuition is through the notion of Property $\hat{P}(j,k)$, which we define formally in what follows.

A *j-vector* $\vec{a} = (a_1, \dots, a_j)$ is a vector where $a_i \in \{-,+\}$ for every $i \in \{1, \dots, j\}$. We denote by $\vec{a}^c = (a_1^c, \dots, a_j^c)$ the j-vector where $a_i^c \neq a_i$ for every $i \in \{1, \dots, j\}$. Let $J = \{v_1, \dots, v_j\}$ be a set of j vertices of $V(\overrightarrow{G})$. Then we define the set

$$N^{\vec{a}}(J) = \left\{ v \in V(\overrightarrow{G}) : v \in N^{a_i}(v_i) \text{ for all } 1 \leq i \leq j \right\} \cup \left\{ v \in V(\overrightarrow{G}) : v \in N^{a_i^c}(v_i) \text{ for all } 1 \leq i \leq j \right\}.$$

Observe that $N^{\vec{a}}(J) = N^{\vec{a}^c}(J)$. We say that $\overrightarrow{G}$ *has Property* $\hat{P}(j,k)$ if for every j-vector $\vec{a}$ and every j-set J we have $|N^{\vec{a}}(J)| \geq k$.

A bijective homomorphism whose inverse is also a homomorphism is an *isomorphism*. Recall that an *automorphism* is an isomorphism of an object to itself. An oriented graph $\overrightarrow{G}$ is *vertex-transitive* if for every two vertices $u, v \in V(\overrightarrow{G})$ there is an automorphism f of $\overrightarrow{G}$ such that $f(u) = v$. We also say that $\overrightarrow{G}$ is *arc-transitive* if given any two arcs $uv, xy \in A(\overrightarrow{G})$ it is possible to find an automorphism f of $\overrightarrow{G}$ such that $f(u) = x$ and $f(v) = y$.

In the context of oriented homomorphisms and pushable homomorphisms, *Paley tournaments* stand, due to their very regular structure, as good candidates to map families of oriented graphs to. In this work, our upper bounds in Theorems 1.4 and 1.5 are actually obtained via pushable homomorphisms to $\overrightarrow{\mathrm{Pal}_7}$, the Paley tournament on seven vertices. $\overrightarrow{\mathrm{Pal}_7}$ (depicted in Figure 1) is the oriented graph (tournament) with vertex set $\mathbb{Z}/7\mathbb{Z} = \{0, 1, \dots, 6\}$ in which ij is an arc if and only if $j - i$ is a nonzero square in $\mathbb{Z}/7\mathbb{Z}$ (where, here and further, all operations involving vertices of $\overrightarrow{\mathrm{Pal}_7}$ are understood modulo 7). In other words, ij is an arc if and only if $j - i \in \{1, 2, 4\}$.

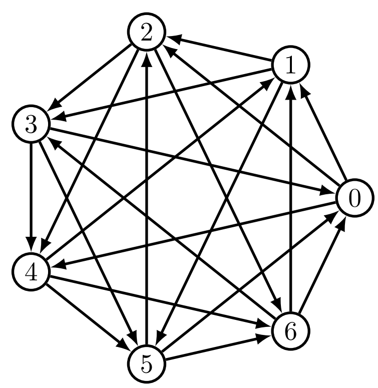


Figure 1: The Paley tournament $\overrightarrow{\mathrm{Pal}_7}$ on seven vertices.

In this work, we will make use of the following properties of interest of $\overrightarrow{\mathrm{Pal}_7}$.

Lemma 2.1 (Marshall [13]). *$\overrightarrow{\mathrm{Pal}_7}$ is vertex-transitive and arc-transitive.*

Lemma 2.2 (Marshall [13]). *$\overrightarrow{\mathrm{Pal}_7}$ has Properties $\hat{P}(1,6)$ and $\hat{P}(2,2)$.*

We also note the following other interesting properties of $\overrightarrow{\mathrm{Pal}_7}$, which, in brief words, mean that if we start from any vertex u of $\overrightarrow{\mathrm{Pal}_7}$ and look at all the vertices v we can reach through a 2-path or 3-path with some given orientation, then v can essentially be any vertex, except in one peculiar case.

Observation 2.3. *For every $i \in V(\overrightarrow{\mathrm{Pal}_7})$ and $\alpha, \beta, \gamma \in \{+, -\}$, we have*

$$N^\alpha(N^\beta(i)) = \begin{cases} V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{i\} & \text{if } \alpha = \beta \\ V(\overrightarrow{\mathrm{Pal}_7}) & \text{if } \alpha \neq \beta \end{cases} \tag{1}$$

and

$$N^\alpha(N^\beta(N^\gamma(i))) = V(\overrightarrow{\mathrm{Pal}_7}). \tag{2}$$

Proof. As $\overrightarrow{\mathrm{Pal}_7}$ is vertex-transitive, it is enough to verify Equations (1) and (2) for $i = 0$, which can easily be done by hand. □

3 Proofs of Theorems 1.2 and 1.3

Let t be a fixed integer. For a given integer j, we set $f_t(j) = (t-j)(t-2)+1$. In the next result, we show that if an oriented graph $\overrightarrow{G}$ has Property $\hat{P}(t-1, f_t(t-1))$ for some t, then it also has Property $\hat{P}(j, f_t(j))$ for all $j \in \{0, 1, \dots, t-1\}$.

Lemma 3.1. *If an oriented graph $\overrightarrow{G}$ has Property $\hat{P}(t-1, f_t(t-1))$ for some t, then it also has Property $\hat{P}(j, f_t(j))$ for all $j \in \{0, 1, \dots, t-1\}$.*

Proof. Suppose $\overrightarrow{G}$ has Property $\hat{P}(j,k)$. Now consider any $(j-1)$-vector $\vec{a}' = (a_1, \dots, a_{j-1})$, any $(j-1)$-set $J' = \{v_1, \dots, v_{j-1}\}$, and a vertex v_j of $\overrightarrow{G}$ not in J'. Let $\vec{a}_+ = (a_1, \dots, a_{j-1}, +)$, $\vec{a}_- = (a_1, \dots, a_{j-1}, -)$ and $J = \{v_1, \dots, v_{j-1}, v_j\}$. Since $\overrightarrow{G}$ has Property $\hat{P}(j,k)$, we must have $|N^{\vec{a}_+}(J)| \geq k$ and $|N^{\vec{a}_-}(J)| \geq k$. Observe that $N^{\vec{a}_+}(J) \cap N^{\vec{a}_-}(J) = \emptyset$ and that $N^{\vec{a}_+}(J), N^{\vec{a}_-}(J) \subseteq N^{\vec{a}'}(J')$. Thus $\overrightarrow{G}$ has Property $\hat{P}(j-1, 2k)$. We are now done by induction since $f_t(j-1) \leq 2f_t(j)$ for all $j \in \{0, 1, \dots, t-1\}$. □

We now prove the existence of tournaments having Property $\hat{P}(j,k)$ for particular values of j and k.

Lemma 3.2. *For all $t \geq 29$, there exist tournaments with Property $\hat{P}(t-1, t-1)$ and order*

$$c = (t-3) \cdot (t-1) \cdot 2^{t-1}.$$

Proof. Let $\overrightarrow{C}$ be a random tournament in which every arc is oriented in one way or the other with equal probability $\frac{1}{2}$. We show below that the probability that $\overrightarrow{C}$ does not have Property $\hat{P}(t-1, t-1)$ is strictly less than 1 when $|\overrightarrow{C}| = c = (t-3) \cdot (t-1) \cdot 2^{t-1}$. Let $\mathbb{P}(J, \vec{a})$ denote the probability that the *bad event* $|N^{\vec{a}}(J)| < f_t(t-1) = (t-2)+1$ occurs, where J is a $(t-1)$-set of $\overrightarrow{C}$ and $\vec{a}$ is a $(t-1)$-vector. Then

$$\begin{aligned}
\mathbb{P}(J, \vec{a}) &\leq \sum_{\substack{|S| \leq t-2 \\ S \cap J = \emptyset}} \mathbb{P}(S = N^{\vec{a}}(J)) = \sum_{\substack{|S| \leq t-2 \\ S \cap J = \emptyset}} \prod_{x \in S} \mathbb{P}(x \in N^{\vec{a}}(J)) \cdot \prod_{x \notin J \cup S} \mathbb{P}(x \notin N^{\vec{a}}(J)) \\
&= \sum_{\substack{|S| \leq t-2 \\ S \cap J = \emptyset}} 2 \cdot 2^{-|S||J|} \cdot (1 - 2 \cdot 2^{-|J|})^{c-|J|-|S|} \\
&= \sum_{i=0}^{t-2} \binom{c-(t-1)}{i} \cdot 2^{-i(t-2)} \cdot \left(1 - 2^{-(t-2)}\right)^{c-i-(t-1)} \\
&= \left(1 - 2^{-(t-2)}\right)^c \cdot \sum_{i=0}^{t-2} \frac{c^i}{i!} \cdot 2^{-i(t-2)} \cdot \left(\frac{2^{t-2}}{2^{t-2}-1}\right)^{i+t-1} \\
&< e^{-c2^{-(t-2)}} \cdot \sum_{i=0}^{t-2} c^i \frac{2^{(t-2)(t-1)}}{(2^{t-2}-1)^{i+t-1}} \leq e^{-c2^{-(t-2)}} \cdot \frac{2^{(t-2)(t-1)}}{(2^{t-2}-1)^{t-1}} \cdot \sum_{i=0}^{t-2} c^i \\
&< 2e^{-c2^{-(t-2)}} \cdot \frac{c^{t-1}-1}{c-1} \leq e^{-c2^{-(t-2)}} \cdot c^{t-1}.
\end{aligned}$$

Let $\mathbb{P}(\mathcal{B})$ denote the probability that at least one bad event occurs. To prove the statement it is then enough to show that $\mathbb{P}(\mathcal{B}) < 1$. Let T denote the set of all $(t-1)$-sets of vertices of $\overrightarrow{C}$, and W denote the set of all $(t-1)$-vectors having $+$ in the first coordinate. Note that given any $(t-1)$-vector $\vec{a}$, exactly one of $\vec{a}$ and $\vec{a}^c$ must belong to W. Then

$$\begin{aligned}\mathbb{P}(\mathcal{B}) &\leq \sum_{J\in T}\sum_{\vec{a}\in W}\mathbb{P}(J,\vec{a}) < \binom{c}{t-1}\cdot 2^{t-2}\cdot e^{-c2^{-(t-2)}}\cdot c^{t-1}\\ &< \frac{c^{t-1}}{(t-1)!}\cdot 2^{t-2}\cdot e^{-c2^{-(t-2)}}\cdot c^{t-1} < \frac{2^{t-2}}{(t-1)!}\cdot e^{-c2^{-(t-2)}}\cdot c^{2(t-1)}\\ &< e^{-c2^{-(t-2)}}\cdot c^{2(t-1)} < \left(\frac{(t-3)^2\cdot(t-1)^2\cdot 2^{2(t-1)}}{e^{2(t-3)}}\right)^{t-1} < 1.\end{aligned}$$

In particular, the last inequality follows because $t \geq 29$. This completes the proof. $\square$

We now show that if a tournament $\overrightarrow{C}$ has Property $\hat{P}(j, f_t(j))$ for all $j \in \{1,\ldots,\Delta-1\}$ where $t=\Delta$, then any connected oriented graph with maximum degree Δ and degeneracy $\Delta-1$ admits a pushable homomorphism to $\overrightarrow{C}$ (recall, as mentioned in the introduction, that the notion of degeneracy we refer to here is actually that of the underlying graph).

Lemma 3.3. *Let $\overrightarrow{C}$ be an oriented graph having Property $\hat{P}(j, f_t(j))$ for all $j \in \{1,\ldots,\Delta-1\}$ where $t=\Delta$, and $\overrightarrow{G}$ be a connected oriented graph with maximum degree Δ and degeneracy $\Delta-1$. Then $\overrightarrow{G} \xrightarrow{\text{push}} \overrightarrow{C}$.*

Proof. Let us assume the vertices of $\overrightarrow{G}$ are labeled $v_1,\ldots,v_k$ so that each vertex has at most $\Delta-1$ neighbors with smaller index. For every $l \in \{1,\ldots,k\}$, we denote by $\overrightarrow{G}_l$ the oriented graph induced by the vertices in $\{v_1,\ldots,v_l\}$. We now inductively construct a homomorphism $g : \overrightarrow{G} \to \overrightarrow{C}$ with the following properties:

- For every $l \in \{1,\ldots,k\}$, the partial mapping $g(v_1),\ldots,g(v_l)$ is a homomorphism from $\overrightarrow{G}_l$ to $\overrightarrow{C}$.
- For every $i > l$, all neighbors of v_i with index at most l have different images by the mapping g.

For $l=1$, consider any partial mapping $g(v_1)$. Suppose now that the function g satisfies the above two properties for all $i \leq l$ for some fixed $l \in \{1,\ldots,k-1\}$. Let A be the set of neighbors of v_{l+1} with index greater than $l+1$, and B be the set of vertices with index at most l and with at least one neighbor in A. Note that $|B| \leq (\Delta-2)|A|$.

Let D be the set of possible options for $g(v_{l+1})$ leading to the partial mapping being a homomorphism from $\overrightarrow{G}_{l+1}$ to $\overrightarrow{C}$. Let A' be the set of neighbors of v_{l+1} with index less than $l+1$. Therefore, due to Property $\hat{P}(|A'|, f_t(|A'|))$ of $\overrightarrow{C}$, we have

$$|D| \geq f_t(|A'|) = (\Delta-|A'|)(\Delta-2)+1 > (\Delta-2)|A| \geq |B|,$$

which implies $|D| > |B|$. Thus choose any vertex from $D\setminus B$ as the image $g(v_{l+1})$. Note that the resulting partial mapping satisfies the two required conditions as well. This concludes the proof. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The lower bound follows from Theorem 1.1 and the bound $2^{\Delta/2} \leq \chi_o(\mathcal{G}_\Delta)$ established by Kostochka, Sopena and Zhu in [11]. We now focus on proving the upper bound.

Let $\overrightarrow{G}$ be a connected oriented graph with maximum degree $\Delta \geq 29$. Note that if $\overrightarrow{G}$ is not Δ-regular then $\overrightarrow{G}$ is $(\Delta-1)$-degenerate. Observe that Lemma 3.2 ensures the existence of an oriented graph $\overrightarrow{C}$ on $c = (\Delta-3)\cdot(\Delta-1)\cdot 2^{\Delta-1}$ vertices having Property $\hat{P}(\Delta-1,\Delta-1)$. As $f_\Delta(\Delta-1) = (\Delta-1)$, Lemma 3.1 implies that $\overrightarrow{C}$ has Property $\hat{P}(j, f_\Delta(j))$ for all $j \in \{1,\ldots,\Delta-1\}$. After that, Lemma 3.3 implies $\overrightarrow{G} \xrightarrow{\text{push}} \overrightarrow{C}$. Thus we are done in this case.

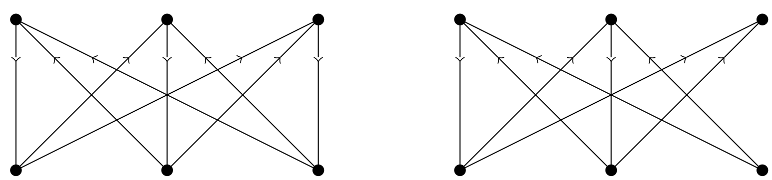
So assume $\overrightarrow{G}$ is Δ-regular. Delete one arc uv from $\overrightarrow{G}$ to obtain a connected oriented graph with maximum degree Δ and degeneracy $\Delta - 1$. This new oriented graph admits a pushable homomorphism g to an oriented graph $\overrightarrow{C}$ with Property $\hat{P}(j, f_t(j))$ for all $j \in \{1, \dots, \Delta - 1\}$. Now add two new vertices x and y to $\overrightarrow{C}$ to obtain a new oriented graph $\overrightarrow{\hat{C}}$. Modify the pushable homomorphism g to $\hat{g}$ by setting $\hat{g}(u) = x$, $\hat{g}(v) = y$ and $\hat{g}(w) = g(w)$ for all $w \neq u, v$. Moreover, choose the direction of the arcs incident with x and y in such a way that $\hat{g}$ is a pushable homomorphism from $\overrightarrow{G}$ to $\overrightarrow{\hat{C}}$. □

The proof of Theorem 1.2 above can also be employed to prove Theorem 1.3.

Proof of Theorem 1.3. The lower bound is due to a result of Kostochka, Sopena and Zhu in [11]. Let us now focus on the upper bound. From Lemmas 3.2 and 3.3, we know that if $\overrightarrow{G}$ has maximum degree Δ and is $(\Delta - 1)$-degenerate, then $\chi_o(\overrightarrow{G}) \leq 2\chi_p(\overrightarrow{G}) \leq 2(\Delta - 3) \cdot (\Delta - 1) \cdot 2^{\Delta}$. Thus we are done for all oriented graphs of $\mathcal{G}_{\Delta}$ but the ones that are Δ-regular. For these oriented graphs, the upper bound can be proved similarly as in the proof of Theorem 1.2. □

4 Proof of Theorem 1.4

The lower bound follows from the existence of subcubic oriented graphs with pushable chromatic number 6, such as the one depicted in Figure 2(i).



(i) (ii)

Figure 2: A cubic oriented graph with pushable chromatic number 6 (i), and an oriented graph with maximum average degree strictly less than 3 and pushable chromatic number 5 (ii).

To prove the upper bound of Theorem 1.4 we show that any subcubic oriented graph $\overrightarrow{G}$ admits a pushable homomorphism to the Paley tournament $\overrightarrow{\mathrm{Pal}_7}$ on seven vertices. We prove this in the rest of this section.

Assume that this does not hold for all subcubic oriented graphs, and consider $\overrightarrow{H}$ a minimum (with respect to its number of vertices) subcubic oriented graph that does not admit a pushable homomorphism to $\overrightarrow{\mathrm{Pal}_7}$. We prove that $\overrightarrow{H}$ cannot contain certain configurations until we finally reach a contradiction to its existence. Note that $\overrightarrow{H}$ must be connected due to the minimality condition.

We first show that $\overrightarrow{H}$ must be cubic and large enough.

Lemma 4.1. *$\overrightarrow{H}$ is cubic.*

Proof. Assume $\overrightarrow{H}$ has a degree-1 vertex u. By minimality, there exists a pushable homomorphism f from $\overrightarrow{H} - \{u\}$ to $\overrightarrow{\mathrm{Pal}_7}$. It is possible to extend f to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$ due to Property $\hat{P}(1, 6)$, a contradiction. Now assume $\overrightarrow{H}$ has a degree-2 vertex u with neighbors v and w. Consider the oriented graph $\overrightarrow{H_1}$ obtained from $\overrightarrow{H}$ by deleting u and adding the arc wv if v and w are not already adjacent. By minimality, there exists a pushable homomorphism f from $\overrightarrow{H_1}$ to $\overrightarrow{\mathrm{Pal}_7}$. It is possible to extend f to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$ due to Property $\hat{P}(2, 2)$, a contradiction. □

Lemma 4.2. *$\overrightarrow{H}$ is not a tournament.*

Proof. Any tournament on four vertices is in a push relationship with one of the following two tournaments (both contained in $\overrightarrow{\mathrm{Pal}_7}$): (i) the induced tournament $\overrightarrow{\mathrm{Pal}_7}[\{0,1,2,4\}]$ and (ii) the induced tournament $\overrightarrow{\mathrm{Pal}_7}[\{1,2,3,4\}]$. Indeed, consider $\overrightarrow{T}$ a tournament different from these two subtournaments of $\overrightarrow{\mathrm{Pal}_7}$. Let a,b,c,d denote the four vertices of $\overrightarrow{T}$. Start by pushing some vertices of $\overrightarrow{T}$ so that all arcs incident to a are out-going from a. If $\overrightarrow{T}[\{b,c,d\}]$ is a directed triangle, then we get $\overrightarrow{\mathrm{Pal}_7}[\{0,1,2,4\}]$ (where a,b,c,d play the role of $0,1,2,3$, respectively). Assume thus that $\overrightarrow{T}[\{b,c,d\}]$ is acyclic. Without loss of generality, we can assume the arcs are bc, bd and cd. In that case, let us further push b and c. The orientation we get is then $\overrightarrow{\mathrm{Pal}_7}[\{1,2,3,4\}]$ (where a,b,c,d play the role of $4,2,3,1$, respectively). Thus, if $\overrightarrow{H}$ is an orientation of K_4 then it must admit a pushable homomorphism to $\overrightarrow{\mathrm{Pal}_7}$, a contradiction. $\square$

Observe that a connected cubic oriented graph $\overrightarrow{G}$ that is not a tournament (i.e., not an orientation of K_4), must have its underlying graph having one of the configurations depicted in Figure 3. Indeed, if G has a triangle $abca$, then either none of ab, bc and ca is shared by another triangle (and we get the configuration in Figure 3(ii)), or $abc'a$ is another triangle, in which case $cc' \notin E(G)$ since G is not K_4 (and we get the configuration in Figure 3(i)). If G has no triangle but has a square $abcda$, then note that we must have $ac \notin E(G)$ and $bd \notin E(G)$ as otherwise we would get a triangle; then we get the configuration in Figure 3(iii). Lastly, if G has neither triangles nor squares, then consider any two adjacent vertices u and v, where the other two other two neighbours of u are u_1,u_2 and the other two neighbours of v are v_1,v_2. Now, if any two vertices in $\{u,v,u_1,u_2,v_1,v_2\}$ are the same, then either G has loops, multiple edges, triangles or squares. So all these vertices must be different; we get the configuration in Figure 3(iv).

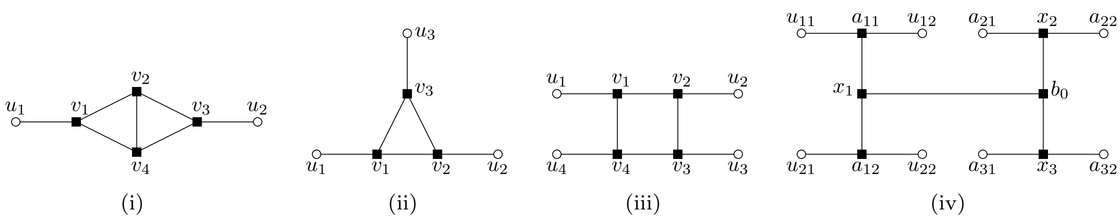


Figure 3: Configurations needed for proving Theorem 1.4. Black square vertices are distinct vertices whose full neighborhood is part of the configuration. White circle vertices are (not necessarily distinct) vertices that might have other neighbors outside the configuration.

In what follows, we prove that none of these configurations can be present in $\overrightarrow{H}$, and thus that $\overrightarrow{H}$ cannot exist. We first introduce some notation and raise some remarks. We below deal with *partial matrices*, i.e., matrices whose entries are either empty or contain an element of $V(\overrightarrow{\mathrm{Pal}_7})$. The ij^{th} entry of a matrix X is denoted by $X(i,j)$. Given two matrices X_1 and X_2 of the same dimension, the matrices $X_1 \pm X_2$ are well defined by setting $(X_1 \pm X_2)(i,j) = X_1(i,j) \pm X_2(i,j)$, with the convention that $\emptyset \pm x = x \pm \emptyset = \emptyset$ for every entry x.

In the upcoming lemmas, we will often use implicitely the following observation to check the correctness of some extensions of pushable homomorphisms : if, for some (i,j), we have $(X_2 - X_1)(i,j) \in \{1,2,4\}$, then taking $f(u) = X_1(i,j)$ and $f(v) = X_2(i,j)$ defines a homomorphism of the arc uv to $\overrightarrow{\mathrm{Pal}_7}$. Similarly, if $(X_2 - X_1)(i,j) \in \{3,5,6\}$ for some (i,j), then taking $f(u) = X_1(i,j)$ and $f(v) = X_2(i,j)$ defines a homomorphism of the arc vu to $\overrightarrow{\mathrm{Pal}_7}$.

Lemma 4.3. *The configuration depicted in Figure 3(i) cannot be contained in H.*

Proof. Assume that H contains the configuration depicted in Figure 3(i). Let $\overrightarrow{H_1}$ be the oriented graph obtained from $\overrightarrow{H}$ by deleting the vertices in $\{v_1,v_2,v_3,v_4\}$, and let $\overrightarrow{H_2}$ be the oriented graph obtained from $\overrightarrow{H}$ by deleting the arc between u_2 and v_3. By minimality, $\overrightarrow{H_1}$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. Up to pushing vertices in $\overrightarrow{H_1}$, we can assume that f is actually an oriented homomorphism.

As $\overrightarrow{\mathrm{Pal}_7}$ is vertex-transitive, without loss of generality we can assume that $f(u_1) = 0$. Moreover, up to pushing v_1, v_2 and v_4, we can assume that $\overrightarrow{H}$ has the arcs u_1v_1, v_2v_1, v_4v_1. Furthermore, up to exchanging v_2 and v_4 and then pushing v_3, we can also assume that $\overrightarrow{H_2}$ has the arcs v_2v_4 and v_3v_2.

We show that for every $\ell \in \{1, \dots, 6\}$ we can extend f to an oriented homomorphism from $\overrightarrow{H_2}$ to $\overrightarrow{\mathrm{Pal}_7}$ satisfying $f(v_3) = \ell$. This allows to conclude: let $\beta \in \{+, -\}$ such that v_3 is a β-neighbor of u_2 in $\overrightarrow{H}$ after having possibly pushed v_3 to obtain the arc v_3v_2. Since $|N^\beta(f(u_2))| = 3$, there exists $\ell \in N^\beta(f(u_2)) \setminus \{0\}$. We then extend f to an oriented homomorphism from $\overrightarrow{H_2}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $f(v_3) = \ell$. Due to the choice of ℓ, this is also an oriented homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$, a contradiction.

Therefore, in order to prove that $\overrightarrow{H}$ cannot contain the configuration depicted in Figure 3(i), it only remains to extend f to an oriented homomorphism from $\overrightarrow{H_2}$ to $\overrightarrow{\mathrm{Pal}_7}$ satisfying $f(v_3) = \ell$ for every $\ell \in \{1, \dots, 6\}$. To this end, we consider the following matrices:

$$X_{v_1} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}, \; X_{v_2} = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}, \; X_{v_4} = \begin{bmatrix} 4 & 5 & 0 \\ 0 & 1 & 3 \\ 6 & 0 & 2 \end{bmatrix},$$

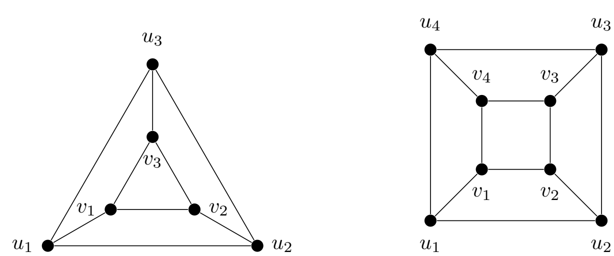
and

$$X^+_{v_3} = \begin{bmatrix} 6 & 0 & 2 \\ 4 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \; X^-_{v_3} = \begin{bmatrix} 3 & 4 & 6 \\ 5 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}.$$

Let $\alpha \in \{+, -\}$ such that $v_3 \in N^\alpha(v_4)$, and $\ell \in \{1, \dots, 6\}$. Observe that all values $\{1, \dots, 6\}$ are present in the matrix $X^\alpha_{v_3}$, hence we can take $(i, j) \in \{1, 2, 3\}^2$ such that $\ell = X^\alpha_{v_3}(i, j)$. We can then extend f to an oriented homomorphism from $\overrightarrow{H_2}$ to $\overrightarrow{\mathrm{Pal}_7}$ by choosing $f(v_k) = X_{v_k}(i, j)$ for all $k \in \{1, 2, 4\}$ and $f(v_3) = X^\alpha_{v_3}(i, j) = \ell$, which concludes the proof. $\square$

Before moving on to proving Lemma 4.5, we need to show the following.

Lemma 4.4. *The graph T_3 depicted in Figure 4(i) cannot be contained in H.*



(i) The graph T_3 (ii) The graph T_4

Figure 4: Two cubic graphs mentioned in the proof of Theorem 1.4.

Proof. Since T_3 is cubic and H is connected, if H contains T_3 then $H = T_3$. Therefore, it is enough to show that for any orientation $\overrightarrow{T_3}$ of T_3, there exists a pushable homomorphism f from $\overrightarrow{T_3}$ to $\overrightarrow{\mathrm{Pal}_7}$. Note that regardless of the orientation $\overrightarrow{T_3}$, it is always possible to push some vertices among $\{u_1, u_2, u_3\}$ so that $\overrightarrow{T_3}$ the vertices in $\{u_1, u_2, u_3\}$ induce a directed cycle. Indeed, if $\overrightarrow{T_3}[\{u_1, u_2, u_3\}]$ is an acyclic tournament with, say, source u_1 and sink u_3, then pushing u_1 and u_3 gives what we want. Moreover, we can also push some of the vertices among $\{v_1, v_2, v_3\}$ to obtain the arcs u_1v_1, u_2v_2, u_3v_3 as well (just push every vertex v_i whose incident arc to u_i does not have the desired direction).

Up to relabelling the vertices of $\overrightarrow{T_3}$, we can assume that some of its vertices were pushed so that we have the arcs u_1u_2, u_2u_3 and u_3u_1. We now define a homomorphism from $\overrightarrow{T_3}$ to $\overrightarrow{\mathrm{Pal}_7}$. We first set $f(u_1) = 1$, $f(u_2) = 2$ and $f(u_3) = 4$. As shown in Figure 5, whatever the orientation of the triangle

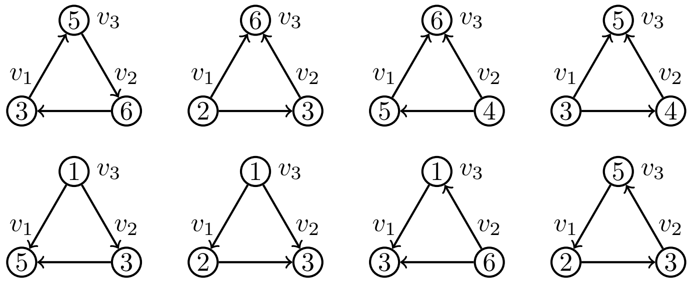


Figure 5: The 8 orientations of the inner cycle induced by $\{v_1, v_2, v_3\}$.

induced by $\{v_1, v_2, v_3\}$ in $\overrightarrow{T_3}$ is, we are always able to choose $f(v_1) \in \{2, 3, 5\}$, $f(v_2) \in \{3, 4, 6\}$ and $f(v_3) \in \{1, 5, 6\}$ to extend f to an oriented homomorphism from $\overrightarrow{T_3}$ to $\overrightarrow{\mathrm{Pal}_7}$. $\square$

Lemma 4.5. *The configuration depicted in Figure 3(ii) cannot be contained in H.*

Proof. Assume that H contains the configuration depicted in Figure 3(ii). Due to Lemmas 4.3 and 4.4, we can assume that u_1 and u_2 are distinct non-adjacent vertices. Moreover, it is possible to push some of the vertices among $\{v_1, v_2\}$ to make sure that $\overrightarrow{H}$ has the arcs u_1v_1, u_2v_2. Furthermore, by symmetry, we can assume the arc v_1v_2 is present in $\overrightarrow{H}$.

Let $\overrightarrow{H_1}$ be the oriented graph obtained by adding the arc u_1u_2 in $\overrightarrow{H}$. We also denote by $\overrightarrow{H_2}$ the oriented graph obtained from $\overrightarrow{H_1}$ by deleting the vertices v_1, v_2 and v_3. By minimality, $\overrightarrow{H_2}$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. Up to replacing $\overrightarrow{H_2}$ (together with $\overrightarrow{H_1}$ and $\overrightarrow{H}$) by a push-equivalent oriented graph, we can assume that f is an oriented homomorphism. However, note that this may cause the arc u_1u_2 to be reversed in $\overrightarrow{H_2}$. This occurs if we needed to push u_1 or u_2 in $\overrightarrow{H_2}$ in order to make f an oriented homomorphism. We again push (if needed) v_1 and v_2 to obtain the arcs u_1v_1 and u_2v_2 in $\overrightarrow{H}$. Observe that u_1u_2 and v_1v_2 are both present in $\overrightarrow{H}$ or both reversed. By symmetry, we can only consider the first case. Finally, up to pushing v_3, we assume that the arc v_3v_1 is in $\overrightarrow{H}$.

Let $\overrightarrow{H_3}$ be the oriented graph obtained from $\overrightarrow{H}$ by deleting the arc between u_3 and v_3. Similarly to the proof of Lemma 4.3, we first extend f to an oriented homomorphism from $\overrightarrow{H_3}$ to $\overrightarrow{\mathrm{Pal}_7}$, with some additional constraint on $f(v_3)$, and then extend f to $\overrightarrow{H}$.

As $\overrightarrow{\mathrm{Pal}_7}$ is arc-transitive, without loss of generality we can assume that $f(u_1) = 0$ and $f(u_2) = 1$. Now consider the following matrices:

$$X_{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \ X_{v_2} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \\ 5 \end{bmatrix}, \ X_{v_3}^{+} = \begin{bmatrix} 4 \\ 0 \\ 6 \\ 5 \\ 2 \end{bmatrix}, \ X_{v_3}^{-} = \begin{bmatrix} 0 \\ 6 \\ 4 \\ 1 \\ 3 \end{bmatrix}.$$

Let $\alpha, \beta \in \{+, -\}$ such that v_3 is an α-neighbor of v_2 and a β-neighbor of u_3. Let S_α be the set of all integers appearing in at least one entry in $X_{v_3}^{\alpha}$. Observe that for every $\ell \in S_\alpha$, there exists $j \in \{1, \dots, 5\}$ such that $\ell = X_{v_3}^{\alpha}(1, j)$. By choosing $f(v_k) = X_{v_k}(1, j)$ for $k \in \{1, 2\}$ and $f(v_3) = \ell$, we can extend f to an oriented homomorphism from $\overrightarrow{H_3}$ to $\overrightarrow{\mathrm{Pal}_7}$ satisfying $f(v_3) = \ell$.

Observe that $|S_\alpha| = 5$. Hence, since $|N^{\beta}(f(u_2))| = 3$, we can choose an $\ell \in N^{\beta}(f(u_2)) \cap S_\alpha$. The corresponding extension of f is now an oriented homomorphism from $\overrightarrow{H_1}$, to $\overrightarrow{\mathrm{Pal}_7}$. Since $\overrightarrow{H}$ is a subgraph of $\overrightarrow{H_1}$, we obtain a contradiction. Therefore, $\overrightarrow{H}$ cannot contain the configuration depicted in Figure 3(ii). $\square$

The proof of Lemma 4.7 is similar to Lemma 4.5. In particular, we first prove an auxiliary lemma in the spirit of Lemma 4.4.

Lemma 4.6. *The graph H cannot be the graph T_4 depicted in Figure 4(ii).*

Proof. It is enough to show that for any orientation $\overrightarrow{T_4}$ of the cubic graph T_4 depicted in Figure 4(ii), there exists a pushable homomorphism f from $\overrightarrow{T_4}$ to $\overrightarrow{\mathrm{Pal}_7}$. We consider two cases, depending on parity of the number of backward arcs of the 4-cycles of $\overrightarrow{T_4}$.

Let us first make more precise what are the backward and forward arcs (of an oriented cycle) we are going to refer to. Here and further, we deal with cycles in some oriented graphs whose underlying graph is depicted in some figure. For each such cycle C, the depicted embedding defines a natural clockwise ordering of its vertices. With respect to that ordering, for any two adjacent vertices u and v of C where, say, u precedes v, in the orientation of C we say that the arc between u and v is *forward* if uv is the arc, while we say that the arc is *backward* if vu is the arc.

Case 1: Suppose that $\overrightarrow{T_4}$ contains a 4-cycle with an odd number of backward arcs. By symmetry, assume that this cycle is the outer one. We can now push some vertices among $\{u_1, u_2, u_3, u_4\}$ to make sure that $\overrightarrow{T_4}$ has the arcs u_1u_2, u_1u_4, u_4u_3 and u_3u_2. Up to pushing some vertices among $\{v_1, v_2, v_3, v_4\}$, we can also assume that $\overrightarrow{T_4}$ contains all the arcs u_iv_i for $i \in \{1, 2, 3, 4\}$. Let $f(u_1) = 0, f(u_2) = 4, f(u_3) = 2$ and $f(u_4) = 1$. As shown in Figure 6, whatever the orientation of the 4-cycle induced by $\{v_1, v_2, v_3, v_4\}$ is, we are always able to choose $f(v_1) \in \{1, 2, 4\}$, $f(v_2) \in \{1, 5, 6\}$, $f(v_3) \in \{3, 4, 6\}$ and $f(v_4) \in \{2, 3, 5\}$ to extend f to a homomorphism from $\overrightarrow{T_4}$ to $\overrightarrow{\mathrm{Pal}_7}$.

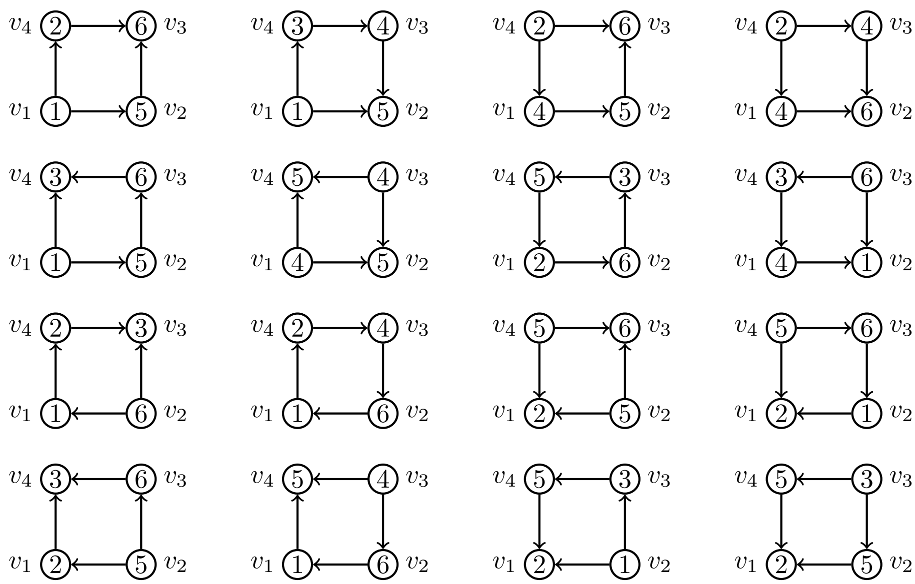


Figure 6: The 16 possible orientations of the inner cycle induced by $\{v_1, v_2, v_3, v_4\}$.

Case 2: Suppose that every 4-cycle of $\overrightarrow{T_4}$ has an even number of backward arcs. Up to pushing some vertices among $\{u_1, u_2, u_3, u_4\}$, we can assume that $\overrightarrow{T_4}$ has the arcs u_1u_2, u_1u_4, u_3u_2 and u_3u_4. We also push some vertices among $\{v_1, v_2, v_3, v_4\}$ such that $\overrightarrow{T_4}$ contains all the arcs u_iv_i for $i \in \{1, 2, 3, 4\}$. By hypothesis, observe that $\overrightarrow{T_4}$ must contain the arcs v_1v_2, v_1v_4, v_3v_2 and v_3v_4. We can then define an oriented homomorphism from $\overrightarrow{T_4}$ to $\overrightarrow{\mathrm{Pal}_7}$ by setting $f(u_1) = f(u_3) = 0$, $f(v_1) = f(v_3) = f(u_2) = f(u_4) = 1$ and $f(v_2) = f(v_4) = 2$. □

Using this auxiliary lemma, we can prove that H does not contain the next configuration.

Lemma 4.7. *The configuration depicted in Figure 3(iii) cannot be contained in H.*

Proof. Assume that H contains the configuration depicted in Figure 3(iii). We follow the same approach as in Lemma 4.5: we first use Lemma 4.6 and symmetry to show we can add the are u_1u_2. Then we remove some vertices and use minimality to obtain a pushable homomorphism, that we extend in two steps to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$.

We first assume that u_1 and u_2 are distinct non-adjacent vertices due to Lemmas 4.5 and 4.6. Up to renaming the vertices of the graph, we assume that if u_3, u_4 are adjacent, then either u_1, u_4 are adjacent or u_2, u_3 are adjacent. Indeed, if u_3 and u_4 are adjacent while all the other pairs are not, then we can relabel vertices and "rotate" the configuration so that e.g. u_2 and u_3 only are adjacent. Moreover, up to pushing v_1 or v_2, we can assume that $\overrightarrow{H}$ has the arcs v_1u_1, v_2u_2. Furthermore, by symmetry, we can assume the arc v_2v_1 is present in $\overrightarrow{H}$.

Let $\overrightarrow{H_1}$ be the oriented graph obtained from $\overrightarrow{H}$ by adding the arc u_1u_2 and an arc between u_3 and u_4 (if such is not already present) in such a way that the oriented cycle induced by the vertices $\{u_3, v_3, v_4, u_4\}$ has an even number of forward and backward arcs.

After that, let $\overrightarrow{H_2}$ be the oriented graph obtained from $\overrightarrow{H_1}$ by deleting the set of vertices $\{v_1, v_2, v_3, v_4\}$, and $\overrightarrow{H_3}$ be the oriented graph obtained from $\overrightarrow{H}$ by deleting the arc between u_3 and v_3, and the arc between u_4 and v_4.

By minimality, $\overrightarrow{H_2}$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. Again, up to replacing $\overrightarrow{H_2}$ (together with $\overrightarrow{H_1}$ and $\overrightarrow{H}$) by a push-equivalent oriented graph, we can assume that f is an oriented homomorphism. Observe that, because we have replaced $\overrightarrow{H_2}$ by a push-equivalent oriented graph, the arc joining u_1 and u_2 may have its direction changed compared to what it was before the replacement. We push v_1 or v_2 if needed to make sure that $\overrightarrow{H_2}$ contains v_1u_1 and v_2u_2. Now either $\overrightarrow{H_2}$ contains both u_1u_2 and v_2v_1 or both u_2u_1 and v_1v_2. By symmetry, we consider only the first case. Moreover, since $\overrightarrow{\mathrm{Pal}_7}$ is arc-transitive, we can assume that $f(u_1) = 0$ and $f(u_2) = 1$.

Up to pushing v_3 or v_4 if needed, we can assume that $\overrightarrow{H_1}$ contains the arcs v_1v_4 and v_2v_3. Now consider the following matrices:

$$X_{v_1} = \left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 3 & 5 & 6 \\ 3 & 5 & 6 & 3 & 5 & 6 \\ 3 & 5 & 6 & 3 & 5 & 6 \\ \hline 2 & 2 & 4 & 2 & 4 & 4 \\ 2 & 2 & 4 & 2 & 4 & 4 \\ 2 & 2 & 4 & 2 & 4 & 4 \end{array}\right], \quad X_{v_2} = \left[\begin{array}{ccc|ccc} 6 & 4 & 4 & 5 & 2 & 3 \\ 6 & 4 & 4 & 5 & 2 & 3 \\ 6 & 4 & 4 & 5 & 2 & 3 \\ \hline 4 & 6 & 6 & 5 & 2 & 3 \\ 4 & 6 & 6 & 5 & 2 & 3 \\ 4 & 6 & 6 & 5 & 2 & 3 \end{array}\right], \quad X_{v_4} = \left[\begin{array}{ccc|ccc} 4 & 6 & 0 & 4 & 6 & 0 \\ 5 & 0 & 1 & 5 & 0 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ \hline 5 & 5 & 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 2 \\ 1 & 1 & 3 & 1 & 3 & 3 \end{array}\right],$$

$$X^+_{v_3} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 5 & \emptyset & 1 & 2 \\ 1 & 6 & 5 & 4 & \emptyset & \emptyset \\ \hline 6 & 0 & 1 & \emptyset & 1 & 2 \\ 1 & 1 & 3 & 4 & \emptyset & 6 \\ 5 & 3 & 0 & 3 & 5 & \emptyset \end{array}\right], \quad X^-_{v_3} = \left[\begin{array}{ccc|ccc} 0 & 5 & 6 & 3 & 5 & 6 \\ 1 & 5 & 6 & 1 & 5 & 6 \\ 3 & 5 & 6 & 3 & 5 & 6 \\ \hline 1 & 3 & \emptyset & 4 & 5 & 6 \\ 5 & 3 & 1 & 3 & 0 & 1 \\ 6 & 0 & 1 & 4 & 1 & 2 \end{array}\right].$$

Let $\overrightarrow{H_3}$ be the oriented graph obtained by deleting the arc between u_3 and v_3 and the arc between u_4 and v_4 in $\overrightarrow{H_1}$. For every $\ell \in \{0, \dots, 6\}$, we first extend f to a pushable homomorphism from $\overrightarrow{H_3}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $f(v_3) = \ell$. We then apply, in $\overrightarrow{H_1}$, this homomorphism with the right value of ℓ to obtain a pushable homomorphism from $\overrightarrow{H_1}$ to $\overrightarrow{\mathrm{Pal}_7}$. Recall that $\overrightarrow{H_1}$ and $\overrightarrow{H_3}$ are indeed defined over the same set of vertices, so f is well defined in $\overrightarrow{H_1}$.

Let $\alpha \in \{+, -\}$ such that v_3 is an α-neighbor of v_4 and let $\ell \in \{0, \dots, 6\}$. Observe that there exists $i, j \in \{1, \dots, 6\}$ such that $\ell = X^\alpha_{v_3}(i, j)$. We push the vertex v_1 (resp. v_2) in $\overrightarrow{H_3}$ if $i > 3$ (resp. $j > 3$). Now, if we choose $f(v_k) = X_{v_k}(i, j)$ for all $k \in \{1, 2, 4\}$ and any $f(v_3) = \ell$, then f is a pushable homomorphism from $\overrightarrow{H_3}$ to $\overrightarrow{\mathrm{Pal}_7}$.

It now remains to choose the value of ℓ to extend this homomorphism to $\overrightarrow{H_1}$. Let $\beta_3, \beta_4 \in \{+, -\}$ such that v_3 is a β_3-neighbor of u_3, and v_4 is a β_4-neighbor of u_4 in $\overrightarrow{H_1}$. Consider the set of integers

$$S^\alpha = \left(\bigcup_{\{(i,j) : X_{v_4}(i,j) \in N^{\beta_4}(f(u_4))\}} \{X^\alpha_{v_3}(i, j)\} \right) \setminus \{\emptyset\}.$$

Observe that S^α depends on the value of $f(u_4)$ and β_4.

Intuitively speaking, S^α is the set of those values such that if any of them is assigned as $f(v_3)$, then it is possible to extend f to a homomorphism from $\overrightarrow{H_4}$ to $\overrightarrow{\mathrm{Pal}_7}$, where $\overrightarrow{H_4}$ is the oriented graph obtained from $\overrightarrow{H_1}$ by deleting the arc between u_3 and v_3.

To make this point clearer, let us demonstrate a sample calculation of a such set S^α. Let us assume that $\alpha = +$, $\beta_4 = +$, and $f(u_4) = 0$. Therefore, $N^{\beta_4}(f(u_4)) = N^+(0) = \{1, 2, 4\}$. So we look for those (i, j) for which the set $X_{v_4}(i, j)$ contains $1, 2$ or 4. In this particular case, those indices (i, j) are the following: $(1, 1)$, $(1, 4)$, $(2, 3)$, $(2, 6)$, $(3, 2)$, $(3, 5)$, $(5, 3)$, $(5, 5)$, $(5, 6)$, $(6, 1)$, $(6, 2)$, $(6, 4)$. Now we get S^α by taking the union of all the entries of $X^+_{v_3}$ which have these indices, which, here, is

$$S^\alpha = \{1, 0, 5, 2, 6, 3, 6, 5, 3, 3\} = \{0, 1, 2, 3, 5, 6\} = V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{4\}.$$

Hence, for each $\ell \in S^\alpha$ there exists an extension of f to a pushable homomorphism from $\overrightarrow{H_3}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $f(v_4) \in N^{\beta_4}(f(u_4))$ and $f(v_3) = \ell$. In particular, if ℓ also lies in $N^{\beta_3}(f(u_3))$, then f is also a pushable homomorphism from $\overrightarrow{H_1}$ to $\overrightarrow{\mathrm{Pal}_7}$. Therefore, we can conclude as soon as $N^{\beta_3}(f(u_3)) \cap S^\alpha \neq \emptyset$.

Note that if $|S^\alpha| \geq 5$, then we must have $N^{\beta_3}(f(u_3)) \cap S^\alpha \neq \emptyset$ as $|N^{\beta_3}(f(u_3))| = 3$. Therefore, we can assume that $|S^\alpha| < 5$, which may happen for some values of $f(u_4)$ and β_4. To handle those instances, we split the rest of the proof into two cases. In each of them, we modify f so that $N^{\beta_3}(f(u_3)) \cap S^\alpha \neq \emptyset$ afterwards, which allows to extend f to a pushable homomorphism from $\overrightarrow{H_1}$ to $\overrightarrow{\mathrm{Pal}_7}$. Since $\overrightarrow{H}$ is a subgraph of $\overrightarrow{H_1}$, this leads to a contradiction. Therefore, H cannot contain the configuration depicted in Figure 3(iii).

Case 1: Suppose that v_3 is a $-$-neighbor of v_4, that is, $\alpha = -$ and the arc v_3v_4 is present. In this case, $|S^-| < 5$ only if $f(u_4) = 1$ and $\beta_4 = -$, and we have $S^- = \{0, 3, 5, 6\}$. The only way of having $N^{\beta_3}(f(u_3)) \cap S^- = \emptyset$ happens when $N^{\beta_3}(f(u_3)) = \{1, 2, 4\}$, which implies $f(u_3) = 0$ and $\beta_3 = +$. Since $f(u_3) = 0$ and $f(u_4) = 1$, $\overrightarrow{H_1}$ contains the arc u_3u_4. This means that the oriented cycle induced by $\{u_3, v_3, v_4, u_4\}$ has an odd number of backward and forward arcs. Since this property is invariant with respect to the push operation, this was also valid when $\overrightarrow{H_1}$ was constructed. Due to this construction, we obtain that u_3 and u_4 were adjacent in H itself. Moreover, by hypothesis, this implies that either u_1 and u_4 are adjacent, or u_2 and u_3 are adjacent.

- If u_1 and u_4 are adjacent, then we must have the arc u_1u_4, since $f(u_1) = 0$ and $f(u_4) = 1$. In this case, since $\overrightarrow{H}$ is cubic, the only neighbors of u_4 are u_1, u_3 and v_4, so we can modify f by setting $f(u_4) = 2$, so that we now have $N^{\beta_3}(f(u_3)) \cap S^- \neq \emptyset$.

- If u_2 and u_3 are adjacent, then we must have the arc u_3u_2, since $f(u_3) = 0$ and $f(u_2) = 1$. Similarly, we modify f by setting $f(u_3) = 4$, so that we now have $N^{\beta_3}(f(u_3)) \cap S^- \neq \emptyset$.

Case 2: Suppose that v_3 is a $+$-neighbor of v_4, that is, $\alpha = +$ and the arc v_4v_3 is present. We have $|S^+| < 5$ only when $(f(u_4), \beta_4) = (1, +), (2, +), (0, -), (1, -)$ or $(6, -)$. In these cases, we respectively have $S^+ = \{0, 3, 5, 6\}, \{0, 1, 5\}, \{0, 1, 5, 6\}, \{0, 1, 2, 4\}$ or $\{0, 1, 3, 6\}$.

For $S^+ = \{0, 1, 5, 6\}$, there exist no $f(u_3)$ and β_3 satisfying $N^{\beta_3}(f(u_3)) \cap S^+ = \emptyset$. Thus it is not possible to have $(f(u_4), \beta_4) = (0, -)$. Also note that in order to satisfy $N^{\beta_3}(f(u_3)) \cap S^+ = \emptyset$, for each value of $(f(u_4), \beta_4) = (1, +), (2, +), (1, -)$ or $(6, -)$, we must have $(f(u_3), \beta_3) = (0, +), (2, +), (0, -)$ or $(6, -)$. Since u_3 and u_4 are adjacent in $\overrightarrow{H_1}$, it is not possible to have $f(u_3) = f(u_4)$. Therefore, there exists $\gamma \in \{+, -\}$ such that $(f(u_4), \beta_4) = (1, \gamma)$, and thus $(f(u_3), \beta_3) = (0, \gamma)$. In particular, $\overrightarrow{H_1}$ contains the arc u_3u_4. This means that the oriented cycle induced by $\{u_3, v_3, v_4, u_4\}$ has an odd number of backward and forward arcs. Since this property is invariant with respect to the push operation, this was again also the case when $\overrightarrow{H_1}$ was constructed. This implies that u_3 and u_4 are adjacent in H itself. By hypothesis, either u_1 and u_4 are adjacent, or u_2 and u_3 are adjacent.

- If u_1 and u_4 are adjacent, then we must have the arc u_1u_4, since $f(u_1) = 0$ and $f(u_4) = 1$. In this case, we modify f by setting $f(u_4) = 2$, so that we now have $N^{\beta_4}(f(u_4)) \cap S^+ \neq \emptyset$.

- If u_2 and u_3 are adjacent, then we must have the arc u_3u_2, since $f(u_3) = 0$ and $f(u_2) = 1$. In this case, we modify f by setting $f(u_3) = 4$, so that now we have $N^{\beta_3}(f(u_3)) \cap S^+ \neq \emptyset$. $\square$

Lemma 4.8. *The configuration depicted in Figure 3(iv) cannot be contained in H.*

Proof. Let X be a graph on four vertices having a degree-3 vertex x with neighbors a_1, a_2 and b. There are no more edges in X, that is, the only edges of X are xa_1, xa_2 and xb. Let $h : V(X) \to V(\overrightarrow{\mathrm{Pal}_7})$ be a function such that $h(a_1) = 0$ and $h(a_2) = 1$. Furthermore let $h^* : V(X) \to V(\overrightarrow{\mathrm{Pal}_7})$ be a function such that $h^*(a_1) = h^*(a_2) = 0$. Note that h and h^* are both functions, not homomorphisms. Also, X is an undirected graph. Later in the proof, we will consider different orientations of X and examine how h or h^* can be extended into a homomorphism.

Now let $\overrightarrow{X}$ be an orientation of X. Moreover, let $\overrightarrow{X'}$ be the orientation of X obtained from $\overrightarrow{X}$ by pushing the vertex x. Without loss of generality assume that the arc a_1x is present in $\overrightarrow{X}$. Therefore $\overrightarrow{X}$ is of one of the following four types below. For each type we list a number of observations following, notably, from the arc-transitivity of $\overrightarrow{\mathrm{Pal}_7}$.

Type 1: $\overrightarrow{X}$ has the arcs a_1x, a_2x and bx. Observe that for any $l \in V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{2, 4, 6\}$ it is possible to extend h to a homomorphism from $\overrightarrow{X}$ or $\overrightarrow{X'}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $h(b) = l$. Moreover, for any $l \in V(\overrightarrow{\mathrm{Pal}_7})$ it is possible to extend h^* to a homomorphism from $\overrightarrow{X}$ or $\overrightarrow{X'}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $h^*(b) = l$.

Type 2: $\overrightarrow{X}$ has the arcs a_1x, a_2x and xb. Observe that for any $l \in V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{0, 1\}$ it is possible to extend h to a homomorphism from $\overrightarrow{X}$ or $\overrightarrow{X'}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $h(b) = l$. Moreover, for any $l \in V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{0\}$ it is possible to extend h^* to a homomorphism from $\overrightarrow{X}$ or $\overrightarrow{X'}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $h^*(b) = l$.

Type 3: $\overrightarrow{X}$ has the arcs a_1x, xa_2 and bx. Observe that for any $l \in V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{1\}$ it is possible to extend h to a homomorphism from $\overrightarrow{X}$ or $\overrightarrow{X'}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $h(b) = l$.

Type 4: $\overrightarrow{X}$ has the arcs a_1x, xa_2 and xb. Observe that for any $l \in V(\overrightarrow{\mathrm{Pal}_7}) \setminus \{0\}$ it is possible to extend h to a homomorphism from $\overrightarrow{X}$ or $\overrightarrow{X'}$ to $\overrightarrow{\mathrm{Pal}_7}$ such that $h(b) = l$.

In the first two cases, we are able to convert both h and h^* into homomorphisms as x is adjacent to a_1 and a_2 through arcs having similar directions (out-going or in-coming) with respect to x. In contrast, for the last two cases, we are able to convert h only into a homomorphism.

Assume now that H contains the configuration depicted in Figure 3(iv). Let $\overrightarrow{H_1}$ be the oriented graph obtained from $\overrightarrow{H}$ by deleting the vertex b_0, $\overrightarrow{H_2}$ be the oriented graph obtained from $\overrightarrow{H_1}$ by deleting the set of vertices $\{x_1, x_2, x_3\}$, and $\overrightarrow{H_3}$ be the oriented graph obtained from $\overrightarrow{H_2}$ by deleting the set of vertices $\{a_{11}, a_{12}\}$. By minimality, $\overrightarrow{H_1}$ admits a pushable homomorphism f_1 to $\overrightarrow{\mathrm{Pal}_7}$. Up to replacing $\overrightarrow{H}$ by a push equivalent oriented graph, we can assume that f_1 is an oriented homomorphism. Let f_2 and f_3 be the restriction of f_1 to $\overrightarrow{H_2}$ and $\overrightarrow{H_3}$, respectively.

In what follows, we say that the vertex x_i *is Type-j*, for some $j \in \{1, 2, 3, 4\}$, if the oriented graph induced by $\{a_{i1}, a_{i2}, x_i, b_0\}$ is the same as the Type-j orientation of $\overrightarrow{X}$ or $\overrightarrow{X'}$ where x_i plays the role of x and b_0 plays the role of b for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$. Note that if x_i is Type-1 (Type-3, respectively), then pushing the vertex b_0 will turn it to Type-2 (Type-4, respectively), and vice versa. Now we want to extend f_2 or f_3 to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$. Note that from the above we know that if x_i is Type-1 then it may forbid at most three values for $f(b_0)$, if x_i is Type-2 then it may forbid at most two values for $f(b_0)$, and if x_i is Type-3 or Type-4 then it may forbid at most one value for $f(b_0)$. Moreover, we can (if necessary) push b_0 to ensure that at most one vertex among $\{x_1, x_2, x_3\}$ is Type-1. Therefore, if any of them is Type-3 or Type-4, then we will be able to extend f_2 to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$.

The only bad situation is to have one Type-1 vertex and two Type-2 vertices among $\{x_1, x_2, x_3\}$. We here consider two cases:

Case 1: If x_1 is Type-1, then we simply push a_{11} and a_{12} to make it Type-2. Then we are able to extend f_3 to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$.

Case 2: Assume that x_2 is Type-1 and x_1, x_3 are Type-2. Without loss of generality assume that $f_1(a_{21}) = 0$ and $f_1(a_{22}) = 1$ (since if $f_1(a_{21}) = f_1(a_{22})$, then x_2 forbids only two values for $f(b_0)$). Therefore, x_2 forbids the values $2, 4, 6$ for $f(b_0)$.

By definition of type 2, x_1 (resp. x_3) forbids the values $f_1(a_{11}), f_1(a_{12})$ (resp. $f_1(a_{31}), f_1(a_{32})$). This is correct because $\overrightarrow{\mathrm{Pal}_7}$ is arc-transitive, and we have noticed in the observations on *Type-2* orientations of X that having $h(a_1) = 0$ and $h(a_2) = 1$ forbid the values $\{0, 1\}$ for b. That is, given a Type-2 orientation of X, if a_1 and a_2 are mapped to distinct vertices of $\overrightarrow{\mathrm{Pal}_7}$, then one can extend the mapping to a homomorphism of $\overrightarrow{X}$ such that b gets mapped to any prescribed vertex distinct from the images of a_1, a_2.

Then, we can extend f_2 to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$, except in the situation where $\{f_1(a_{11}), f_1(a_{12}), f_1(a_{31}), f_1(a_{32})\} = \{0, 1, 3, 5\}$.

For $S \subseteq \{a_{11}, a_{12}\}$, we denote by $\overrightarrow{H_2}(S)$ the graph obtained from $\overrightarrow{H_2}$ by pushing the vertices of S. Note that for each choice of S, we can extend f_3 to an oriented homomorphism f_S from $\overrightarrow{H_2}(S)$ to $\overrightarrow{\mathrm{Pal}_7}$ using Observation 2.3 and the fact that a_{11} and a_{12} are degree-2 vertices in $\overrightarrow{H_2}$. We can even ensure that for every vertex $v \notin S$, we have $f_S(v) = f_2(v)$ and, for $v \in \{a_{11}, a_{12}\}$, $f_{\{a_{11},a_{12}\}}(v) = f_{\{v\}}(v)$. We show that there exists a choice of S such that f_S can be extended to $\overrightarrow{H}(S)$.

Let $S = \{a_{11}, a_{12}\}$, and observe that x_1 has Type-1 in $\overrightarrow{H}(S)$.

- If $(f_1(a_{11}), f_1(a_{12})) = (f_S(a_{12}), f_S(a_{11}))$, then due to the property of Type-1 vertices, x_1 does not forbid the values $f_S(a_{11}) = f_1(a_{12})$ and $f_S(a_{12}) = f_1(a_{11})$ for b_0 anymore. In other words, since, for Type-1 orientations of X, the forbidden values for $f(b)$ were different from 0 and 1 (the images of a_1, a_2), here also the set of forbidden values for b_0 must be different from the images of a_{11}, a_{12}. As these values were only forbidden for $f(b_0)$ (due to x_1) up to this point, after pushing the vertices of S these values are no longer forbidden. Therefore, we can extend f_S to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$.

- If $f_1(a_{11}) \neq f_S(a_{12})$, then we claim that $f_{\{a_{12}\}}$ can be extended to $\overrightarrow{H}(\{a_{12}\})$. Indeed, in $\overrightarrow{H}(\{a_{12}\})$, x_1 is Type-3 or Type-4. In particular, we can extend $f_{\{a_{12}\}}$ unless $f_{\{a_{12}\}}(a_{12}) = f_{\{a_{12}\}}(a_{11})$. However, note that we have $f_S(a_{12}) = f_{\{a_{12}\}}(a_{12})$ and $f_{\{a_{12}\}}(a_{11}) = f_2(a_{11}) = f_1(a_{11})$ by definition of f_S. This is impossible by hypothesis.

- Otherwise, we have $f_1(a_{12}) \neq f_S(a_{11})$. Similarly to the previous item, we claim that $f_{\{a_{11}\}}$ can be extended to $\overrightarrow{H}(\{a_{11}\})$. Indeed, x_1 is again Type-3 or Type-4, so $f_{\{a_{11}\}}$ can be extended unless $f_{\{a_{11}\}}(a_{11}) = f_{\{a_{11}\}}(a_{12})$, which is again impossible by hypothesis.

In each case, we can thus extend f_3 to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$, which concludes the proof. $\square$

5 Proof of Theorem 1.5

The lower bound follows from the existence of oriented graphs with maximum average degree less than 3 and pushable chromatic number 5, such as the one depicted in Figure 2(ii). To prove the upper bound of Theorem 1.5, we show below that every oriented graph with maximum average degree less than 3 admits a pushable homomorphism to the Paley tournament $\overrightarrow{\mathrm{Pal}_7}$ on seven vertices. Towards a contradiction, let us assume that there exists $\overrightarrow{H}$, a minimum (with respect to number of vertices) oriented graph having $\mathrm{mad}(\overrightarrow{H}) < 3$ that does not admit a pushable homomorphism to $\overrightarrow{\mathrm{Pal}_7}$. Note that the underlying graph H of $\overrightarrow{H}$ $\overrightarrow{H}$ must be connected due to the minimality condition. We prove below that, by minimality, none of the configurations depicted in Figure 7 can appear in H.

Lemma 5.1. *None of the configurations (i)-(v) depicted in Figure 7 can be contained in H.*

Proof. (i) Suppose H has a degree-1 vertex u with neighbor v, where u is an α-neighbor of v for $\alpha \in \{+, -\}$. By minimality, $\overrightarrow{H} - \{u\}$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. We can extend f to $\overrightarrow{H}$ by setting

$$f(u) = \begin{cases} f(v) + 1 & \text{if } \alpha = + \\ f(v) + 3 & \text{if } \alpha = -, \end{cases}$$

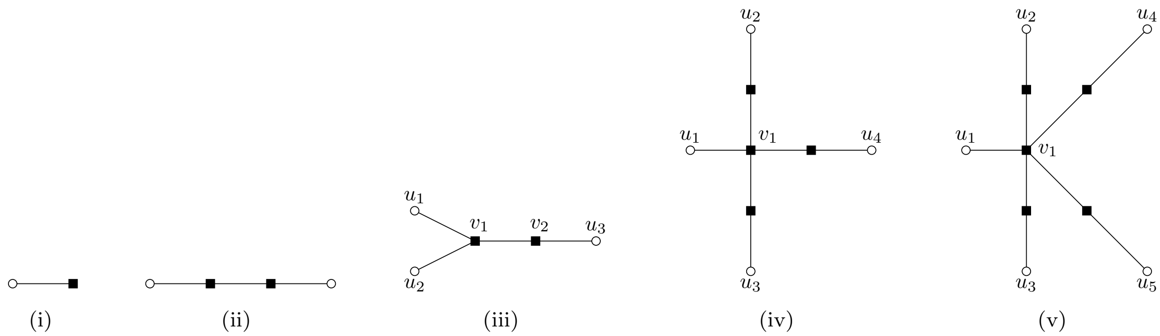


Figure 7: Configurations needed for proving Theorem 1.5. Black square vertices are distinct vertices whose full neighborhood is part of the configuration. White circle vertices are (not necessarily distinct) vertices that might have other neighbors outside the configuration, or be equal.

a contradiction. Therefore, H cannot contain the configuration depicted in Figure 7(i).

(ii) Assume H has two adjacent degree-2 vertices u and v. By minimality, $\overrightarrow{H} - \{u, v\}$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. According to Equation (2) of Observation 2.3, f can be extended to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$, a contradiction. Indeed, denote by u' the other neighbour of u in $\overrightarrow{H}$, and by v' the other neighbour of v. That equation tells us that, whatever $f(u')$ and $f(v')$ are, there is, in $\overrightarrow{\mathrm{Pal}_7}$, a 3-path $f(u')xyf(v')$ with the same orientation as the 3-path from u' to v' through u and v in $\overrightarrow{H}$. Then it suffices to set $f(u) = x$ and $f(v) = y$. Therefore, H cannot contain the configuration depicted in Figure 7(ii).

(iii) Assume H contains the configuration depicted in Figure 7(iii) (where, here and further, we deal with the vertices of a configuration through the notation introduced in the corresponding figure). By minimality, $\overrightarrow{H} - \{v_2\}$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. Suppose that $v_1 \in N^{(\alpha_1,\alpha_2)}(u_1, u_2)$. If $f(v_1) = f(u_3)$, then push v_1 and update $f(v_1)$ to some value in $N^{(\bar{\alpha}_1,\bar{\alpha}_2)}(f(u_1), f(u_2))$ (this is possible since $\overrightarrow{\mathrm{Pal}_7}$ has property $\hat{P}(2, 2)$). Note that the resulting updated f remains a pushable homomorphism. Additionally, now we have $f(v_1) \neq f(u_3)$. It is now possible to extend f to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$ due to Equation (1) in Observation 2.3, a contradiction. Indeed, that equation tells us that, whenever $f(v_1) \neq f(u_3)$, there is always, in $\overrightarrow{\mathrm{Pal}_7}$, a 2-path $f(v_1)xf(u_3)$ with the same orientation as the 2-path from v_1 to u_3 through v_2 in $\overrightarrow{H}$. Then it suffices to set $f(v_2) = x$. Therefore, H cannot contain the configuration depicted in Figure 7(iii).

(iv) – *(v)* Assume H contains one of the configurations depicted in Figure 7(iv) and (v). Let S be the set of black vertices of the configuration. By minimality, $\overrightarrow{H} - S$ admits a pushable homomorphism f to $\overrightarrow{\mathrm{Pal}_7}$. Now, push v_1, if needed, so that at most two of the 2-paths connecting v_1 to the u_i's are directed 2-paths. Note that this indeed holds: pushing v_1 makes each of these directed 2-paths not directed, and *vice versa*. If at least three of the 2-paths are directed, then, when pushing v_1, we make them not directed any more, while the other 2-paths (there is at most one such) get directed.

Without loss of generality, assume that, in the worst-case scenario, u_2 and u_3 are the vertices that are connected by a directed 2-path to v_1. Assume v_1 is an α-neighbor of u_1 for some $\alpha \in \{+, -\}$. Choose a vertex $i \in N^{\alpha}(f(u_1)) \setminus \{f(u_2), f(u_3)\}$ and assign $f(v_1) = i$. Now we are able to extend f to a pushable homomorphism from $\overrightarrow{H}$ to $\overrightarrow{\mathrm{Pal}_7}$ due to Equation (1) in Observation 2.3 (by similar arguments as earlier), a contradiction. Therefore, H cannot contain the configurations depicted in Figure 7(iv) and (v). □

We are now ready to prove Theorem 1.5, which we do using the so-called *discharging method*.

Proof of Theorem 1.5. Let us assign the charge $\mathrm{ch}(v) = d(v)$ to each vertex v of $\overrightarrow{H}$, where, recall, $d(v)$ denotes the degree of v in H, the graph underlying $\overrightarrow{H}$. Since $\mathrm{mad}(\overrightarrow{H}) < 3$, the total sum of the charges is strictly less than $3|V(\overrightarrow{H})|$, that is,

$$\sum_{v \in V(\overrightarrow{H})} \mathrm{ch}(v) < 3|V(\overrightarrow{H})|.$$

Now apply the following discharging procedure: Every vertex v of $\overrightarrow{H}$ with degree at least 4 sends 1/2 to each of its neighbors with degree 2. We show in the following that for every vertex v the resulting charge $\mathrm{ch}^*(v)$ is at least 3, which contradicts the assumption $\mathrm{mad}(H) < 3$. We consider the vertices v accordingly to their degree, which satisfies $d(v) > 1$ by Lemma 5.1(i).

- $d(v) = 2$. Since H does not contain the configurations depicted in Figures 7(ii) and 7(iii), the neighbors of v have degree at least 4 and thus v does not send any charge. Furthermore, v receives exactly $2 \times 1/2 = 1$. Thus, $\mathrm{ch}^*(v) = 2 + 1 = 3$.
- $d(v) = 3$. Since H does not contain the configuration depicted in Figure 7(iii), v does not send any charge. Furthermore, v does not receive any charge. Therefore, we have $\mathrm{ch}^*(v) = 3$.
- $d(v) = 4$. Since H does not contain the configuration depicted in Figure 7(iv), v sends at most $2 \times 1/2 = 1$. Therefore, we have $\mathrm{ch}^*(v) \geq 4 - 1 = 3$.
- $d(v) = 5$. Since H does not contain the configuration depicted in Figure 7(v), v sends at most $3 \times 1/2 = 3/2$. Therefore, we have $\mathrm{ch}^*(v) \geq 5 - 3/2 = 7/2 > 3$.
- $d(v) = k \geq 6$. v sends at most $k \times 1/2 = k/2$ charges. Therefore, we have $\mathrm{ch}^*(v) \geq k - k/2 = k/2 \geq 6/2 = 3$.

Therefore, every vertex v of $\overrightarrow{H}$ gets final charge $\mathrm{ch}^*(v)$ at least 3. Hence

$$3|V(\overrightarrow{H})| > \sum_{v \in V(\overrightarrow{H})} \mathrm{ch}(v) = \sum_{v \in V(\overrightarrow{H})} \mathrm{ch}^*(v) \geq 3|V(\overrightarrow{H})|,$$

since no charge was created after assigning the initial charges, which is a contradiction. Thus every oriented graph with maximum average degree less than 3 admits a pushable homomorphism to $\overrightarrow{\mathrm{Pal}_7}$. $\square$

6 Conclusions and perspectives

In this work, we have studied the pushable chromatic number of several classes of graphs with degree constraints. We have provided bounds for graphs with large maximum degree Δ (Theorem 1.2), graphs with maximum degree $\Delta \leq 3$ (Theorem 1.4), and graphs with maximum average degree less than 3 (Theorem 1.5). None of our results is tight however, and a natural direction for further work could thus be to tighten our bounds. In particular, we wonder whether there exist subcubic graphs or graphs with maximum average degree less than 3 with pushable chromatic reaching the upper bounds we have established. Let us mention that we first checked the proof of Theorem 1.4 through computer programs (before coming up with the presented matrices), and that we did not find any tournament on six vertices for which all configurations in Figure 3 are reducible. Also, although we managed to generate many graphs with maximum average degree less than 3 (planar graphs with girth at least 6, respectively) and check their pushable chromatic number via computer programs, we were not able to spot one with pushable chromatic 6 (5, respectively). These two facts might be good hints regarding the maximum value of the pushable chromatic number of these families of graphs.

Another interesting direction for further research on the topic could be to generalize our results to graphs with given maximum degree Δ more than 3, graphs with given maximum average degree, and planar graphs with given girth. In other words, we wonder how these graph parameters influence the pushable chromatic number. We would be quite interested, for instance, in having bounds for graphs with maximum degree Δ at most 4.

Finally, several recent works have established that, when it comes to coloring, pushable graphs and signed graphs sometimes have very comparable behaviors. Let us recall that a *signed graph* is a graph in which each edge is either positive or negative, and that comes with a vertex-resigning operation which consists in switching the sign of all edges incident with a vertex. It would be interesting to know if, in general, graphs with degree constraints have their pushable chromatic number and signed chromatic number behaving the same. We will propose a study of this very question, inspired from our results in the current work, in a forthcoming paper.

References

[1] J. Bensmail, S. Nandi, and S. Sen. On oriented cliques with respect to push operation. *Discrete Applied Mathematics*, 232:50–63, 2017.

[2] B. Courcelle. The monadic second order logic of graphs VI: On several representations of graphs by relational structures. *Discrete Applied Mathematics*, 54(2):117–149, 1994.

[3] C. Duffy. Homomorphisms of (j, k)-mixed graphs. *PhD Thesis, University of Victoria/University of Bordeaux*, 2015.

[4] C. Duffy. A note on colourings of connected oriented cubic graphs. *arXiv preprint 1908.02883 [cs.DM]*, 2019.

[5] C. Duffy, G. MacGillivray, and É. Sopena. Oriented colourings of graphs with maximum degree three and four. *Discrete Mathematics*, 342(4):959–974, 2019.

[6] D. C. Fisher and J. Ryan. Tournament games and positive tournaments. *Journal of Graph Theory*, 19(2):217–236, 1995.

[7] G. Guégan and P. Ochem. Complexity dichotomy for oriented homomorphism of planar graphs with large girth. *Theoretical Computer Science*, 596:142–148, 2015.

[8] W. F. Klostermeyer. Pushing vertices and orienting edges. *Ars Combinatoria*, 51:65–76, 1999.

[9] W. F. Klostermeyer and G. MacGillivray. Homomorphisms and oriented colorings of equivalence classes of oriented graphs. *Discrete Mathematics*, 274(1–3):161–172, 2004.

[10] W. F. Klostermeyer and Ľ. Šoltés. Hamiltonicity and reversing arcs in digraphs. *Journal of Graph Theory*, 28(1):13–30, 1998.

[11] A. V. Kostochka, É. Sopena, and X. Zhu. Acyclic and oriented chromatic numbers of graphs. *Journal of Graph Theory*, 24:331–340, 1997.

[12] G. MacGillivray and K. L. B. Wood. Re-orienting tournaments by pushing vertices. *Ars Combinatoria*, 57, 2000.

[13] T. H. Marshall. Homomorphism bounds for oriented planar graphs. *Journal of Graph Theory*, 55:175–190, July 2007.

[14] K. M. Mosesian. Strongly basable graphs (in Russian). *Akad. Nauk. Armian. SSR Dokl.*, 54:134–138, 1972.

[15] O. Pretzel. On graphs that can be oriented as diagrams of ordered sets. *Order*, 2:25–40, 1985.

[16] O. Pretzel. On reordering graphs by pushing down maximal vertices. *Order*, 3:135–153, 1986.

[17] O. Pretzel. Orientations and edge functions on graphs. *Surveys in Combinatorics, London Math. Soc. Lecture Notes*, 66:161–185, 1991.

[18] S. Sen. On homomorphisms of oriented graphs with respect to the push operation. *Discrete Mathematics*, 340(8):1986–1995, 2017.

[19] É. Sopena. Homomorphisms and colourings of oriented graphs: An updated survey. *Discrete Mathematics*, 339(7):1993–2005, 2016.